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Hermitian Measures in *W*J*-Algebras in Hilbert Spaces with Conjugation†

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Let $\mathcal{M}$ be a real semifinite W^*-algebra of J-real operators containing no finite central summand in a complex Hilbert space H with conjugation J. Denote by P the quantum logic of all J-orthogonal projections in the von Neumann algebra $\mathcal{N} = \mathcal{M} + i\mathcal{M}$. Let $\mu: P \to R$ be a Hermitian measure. It is shown that there exists an unique J-self-adjoint ultraweakly continuous linear functional ψ on $\mathcal{N}$ such that $\mu(p) = \Re\psi(p)$, $\forall p \in P$.

1. INTRODUCTION

In this paper we continue the description of measures on logics of projections in Hilbert spaces with conjugation (Matvejchuk, 1998).

Let H be a complex Hilbert space with the inner product $(\cdot, \cdot)$. We will denote by S the unit sphere in H. Let J be an operator of *conjugation* in H [i.e., $J^2 = I$; $(Jx, Jy) = (y, x)$, $J(\lambda x + \beta y) = \bar{\lambda}Jx + \bar{\beta}Jy$, $\forall x, y \in H$, $\forall \lambda, \beta \in \mathbf{C}$]. A vector $x \in H$ is said to be *J-real* if $Jx = x$. The vectors $x_{\Re} = 1/2(x + Jx)$ and $x_{\Im} := 1/2i(x - Jx)$ [$= -1/2(ix + J\,ix)$] are J-real, $\forall x \in H$, and $x = x_{\Re} + ix_{\Im}$ The set $H_{\Re}$ of all J-real vectors is a real Hilbert space with respect to the inner product $(\cdot, \cdot)$. Let $S_{\Re}$ denote the set $S \cap H_{\Re}$. Put $\langle x, y\rangle := (Jx, y)$. Let $B \in B(H)$. The operator $B^0 \in B(H)$ such that $\langle Bx, y\rangle = \langle x, B^0y\rangle$, $\forall x, y \in H$, is called a *J-adjoint*. It is clear that $B^0 = JB^*J$ [$= (JBJ)^*$], $(BA)^0 = A^0B^0$, and $A \in B(H)$ is *J-selfadjoint* $\Leftrightarrow A = JA^*J$. An operator $A \in B(H)$ is said to be *J-real* if $JAJ = A$. Note that $A_{J\Re} := 1/2(A + JAJ)$ and $A_{J\Im} := 1/2i(A - JAJ)$ are J-real operators and $A = A_{J\Re} + iA_{J\Im}$.

† This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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A von Neumann algebra $\mathcal{N}$ acting in H is called a W^*J-algebra if $\mathcal{N}$ is closed with respect to the J-adjunction (i.e., $A, \in \mathcal{N}$ implies $A^0 \in \mathcal{N}$). A weakly closed real *-algebra $M \subset B(H)$ with $M \cap iM = \{0\}$ is said to be a *real W^*-algebra*. If M is a real W^*-algebra, then $N := M + iM$ is a von Neumann algebra. Let $\mathcal{N}$ be a W^* J-algebra. It is evident that the set $\mathcal{M}$ of all J-real operators in $\mathcal{N}$ is a real W^*-algebra. $B \in \mathcal{N}$ implies JBJ $[= (B^0)^*]$ $\in \mathcal{N}$. Hence $B_{J\mathfrak{R}}$, $B_{J\mathfrak{I}} \in \mathcal{N}$ and $\mathcal{N} = \mathcal{M} + i\mathcal{M}$.

Let $\mathcal{M}_s$ be the set of all self-adjoint operators in $\mathcal{M}$. Then $\mathcal{M}_s$ is a Jordan algebra (=JW-algebra) with respect to the product $A \circ B := 1/2(AB + BA)$. The $\mathcal{M}_s$ has type I (II, III) $\Leftrightarrow$ the von Neumann algebra $\mathcal{N} = \mathcal{M} + i\mathcal{M}$ has type I (II, III) (Ayupov, 1986).

Let P $[=P(\mathcal{N})]$ denote the set of all J-self-adjoint (= J-orthogonal) projections (= idempotents) from $\mathcal{N}$. With respect to the standard relations, the ordering $p \leq g \Leftrightarrow p = gp$ $(= pg) \Leftrightarrow pH \subseteq gH$, and the orthocomplementation $p \to p^{\perp} := I - p$, the set P is a *quantum logic*. The set of all orthogonal projections in P is denoted by Π $[= \Pi(\mathcal{N})]$. It is clear that $p \in P$ is J-real $\Leftrightarrow$ $p \in \Pi$. Hence $\Pi(B(H))$ is isomorphic to the lattice of all orthogonal projections on $H_{\mathfrak{R}}$.

2. THE STRUCTURE OF THE PROJECTIONS FROM P

Let $p \in P$; then $p^* \in P$. Let p_{or} be the orthogonal projection onto $pH \cap p^*H$. Then $p_{or} \in \Pi$ and p_{or} is the greatest orthogonal projection with the properties $p_{or} \leq p$ (Matvejchuk, 1998). A projection $p \in P$ is said to be *properly skew* projection if $p_{or} = 0$. Let $p \neq p^*$. Then it is clear that $p - p_{or}$ is the properly skew projection. Let $\mathcal{N}^{or}$ denote the set of all orthogonal projections from $\mathcal{N}$.

Let $p \in P$. The positive part of $(p + p^*)$ will be denoted by $(p + p^*)_+$, and by e_+ will be denote the orthogonal projection onto $\overline{(p + p^*)_+ H}$. Then $e_+ pe_+ = \frac{1}{2}(p + p^*)_+$ (Matvejchuk, 1998). Put $e_- := I - e_+$. Now, denote by F_y the orthogonal projection onto $\overline{yH}$, $\forall_y \in B(H)$.

We begin with an important formula on projections from P.

Proposition 1. Let $\mathcal{N}$ be a W^*J-algebra, $p \in P$, and let $e_- pe_+ = w|e_- pe_+|$ be the polar decomposition for $e_- pe_+$. Then $x := e_+ pe_+ (\geq e_+)$ and $v := (1/i)w$ are J-real operators in $\mathcal{N}$, and

$$p = x + iv(x^2 - x)^{1/2} + i(x^2 - x)^{1/2}v^* - v(x - F_x)v^* \qquad (1)$$

Conversely, let $x \in \mathcal{N}$ be an arbitrary J-real operator such that $x \geq F_x$, and let $v \in \mathcal{N}$ be a J-real partial isometry with the initial projection F_x and the final (J-real) one e such that $e \perp F_x$. Then (1) defines a projection in P.

For the proof see Matvejchuk (1998).

To emphasize that p in (1) depends on x and v, we shall use the notation $p(x, v)$ as well. It is easily seen that $\|p(x, v)\| = \|2x - I\| = 2\|x\| - 1$. The projection $p(te, v)$, where $t \in R$ and $e \in \Pi$, is said to be a *simple* projection. Let $S_p := F_x + F_v$, $\forall p = p(x, v) \in P$. Let us denote by P_β the set $\{p(x, v) \in P: \|p(x, v)\| \leq \beta\}$. Let $x = \int \lambda de_\lambda$ be the spectral representation for x (we assume the function $\lambda \to e_\lambda$ to be right continuous). Put

$$x_n := \sum_{i \geq n} \frac{(i + 1/2)}{n} (e_{(i+1)/n} - e_{i/n}) + (e_1 - e_{1-}) \qquad \text{and}$$

$$v_i := v(e_{(i+1)/n} - e_{i/n})$$

Let us mention one consequence of the formula (1).

Corollary 2.

$$p(x_n, v) = \sum_{i \geq n} p\left(\frac{(i + 1/2)}{n} (e_{(i+1)/n} - e_{i/n}), v_i\right) + (e_1 - e_{1-})$$

and $\lim \|p(x, v) - p(x_n, v)\| = 0$.

Obviously, $(\cdot, x)y$ $(\neq 0)$ is a projection $\Leftrightarrow$ $(x, y) = 1$. Let $(\cdot, x)y$ $(\neq 0)$ be a projection. A routine computation shows that:

1. $(\cdot, x)y \in P(B(H)) \Leftrightarrow (\cdot, x)y = (\cdot, Jy^*)y^*$, where $y^* = (y, Jy)^{-1/2}y$.
2. $p_y := (\cdot, Jy)y \in P(B(H)) \Leftrightarrow (y_{\Re}, y_{\Im}) = 0$ and $\|y_{\Re}\|^2 - \|y_{\Im}\|^2 = 1$.
3. Let $p \in P(B(H))$, $p \neq 0$. Then $(\cdot, Jy^*)y^* \leq p$, $\forall y \in pH$: $\|y_{\Re}\| \neq \|y_{\Im}\|$.

Corollary 3. The logic $P(B(H))$ is atomistic.

Remark 4. (i) Let $p = p(x, v) \in P_\beta$ be a properly skew projection, $f \in \mathcal{N}^{or}$, and $f \leq F_p$. Then there exists $g = g(z, w) \in P_\beta$ such that $F_z \leq F_x$, $F_g = f$, $g \leq p$.

(ii) Let $p \in$ II, $f \in \mathcal{N}^{or}$, and $f \leq p$. Then there exists a properly skew projection $g \in P$ such that $F_g = f$, $g \leq P \Leftrightarrow$ there exists $f_0 \in$ II, $f_0 \leq p$, such that $\frac{1}{2} f_0 < f_0 f f_0$ and $v^*v \leq p - f_0$, where v is the partial isometry from the polar representation $f_0^{\perp} f f_0 = v|f_0^{\perp} f f_0|$.

Proof. (i) Let $f \leq F_p$, $f \in$ II. Then $JfJ \leq JF_pJ$. Let $f + JfJ = \int \lambda e_\lambda$ be the spectral representation for $f + JfJ$. Put $e_+ := I - e_1$ and $e_- := I - e_+$. Let $e_- fe_+ = w_o|e_- fe_+|$ be the polar representation for $e_- fe_+$. Then $v_0 = (1/i)w_o$ is a J-reality partial isometry. In addition, $e_+ fe_+ = e_+ JfJe_+$, $y_o := e_+ fe_+$ is a J-real operator, and $y_o > \frac{1}{2} e_+$. Put $x_o = y_o(2y_o - I)^{-1}$ and $g = p(x_o, v_o)$. Then g is the suitable projection.

Lemma 5. Let $\mathcal{N}$ be a countably decomposable von Neumann algebra and let ϕ be a faithful normal state on $\mathcal{N}$. Let $p = p(x, v) \in P_\beta$ be such that the spectrum of x is continuous in $(c, +\infty)$, where $c = \inf\{(x\kappa, \kappa): \kappa \in S \cap$

$e_+H\}$. Then for any natural m there exists a mutually orthogonal family $\{p_i\}_1^m \subset P_\beta$ such that $p = \Sigma\, p_i$ and $\phi(S_{p_i}) = (1/m)\,\phi(S_p)$, $\forall i$.

Proof. Let $x = \int \lambda de_\lambda$ be the spectral representation for x again. By the assumption on x, the function $\phi(e_\lambda + ve_\lambda v^*)$ is continuous on $(c, +\infty)$. Hence there exists $\{e_i\}_1^m \subset \text{II}$ with the properties: (i) $\Sigma_1^m\, e_i = F_x$, (ii) $e_i x = xe_i$, $\forall i$, (iii) $\phi(e_i + ve_i v^*) = (1/m)\phi(S_p)$, $\forall i$. By the construction, $\{p(xe_i, ve_i)\}_1^m$ is a suitable family.

Remark 6. Let $\mathcal{N}$ be a countably decomposable continuous von Neumann algebra. Then assesion of Lemma 5 is true for any $p \in P$.

3. INDEFINITE METRIC SUBSPACES IN *H*

Let $e \in \text{II}$ $(0 < e < I)$. The set $\mathcal{H}_e := eH_{\mathfrak{R}} \oplus ie^\perp H_{\mathfrak{R}}$ is a real Hilbert space with respect to the product $(\cdot, \cdot)$ and $H = \mathcal{H}_e + i\mathcal{H}_e$. Let $\bar{J}$ denote the restriction of J to $\mathcal{H}_e$. Clearly $\bar{J} = (e - e^\perp)/\mathcal{H}_e$ and $\bar{J}$ is a symmetry (i.e., $\bar{J}^2 = I$, $\bar{J} = \bar{J}^*$ in $\mathcal{H}_e$. Consequently, we have:

Every $b \in B(\mathcal{H}_e)$ can be uniquely extended to a bounded linear operator b_H on H, $(b_H)^* = (b^*)_H$, and if p is a bounded projection on $\mathcal{H}_e$, then p_H is a bounded projection, too. In addition, if a projection p is $\bar{J}$-self-adjoint, then p_H is J-self-adjoint. Conversely, if $q \in P$ and $q\mathcal{H}_e \subseteq \mathcal{H}_e$, then $q/\mathcal{H}_e$ is a $\bar{J}$-self-adjoint projection.

With respect to the product $[x, y] := (Jx, y)$, $\forall x, y \in \mathcal{H}_e$, the set $\mathcal{H}_e$ *is a real indefinite metric space and* $\bar{J}$ is a canonical symmetry with respect to the canonical decomposition $\mathcal{H}_e = \mathcal{H}_e^+[\dot{+}]\mathcal{H}_e^-$, where $\mathcal{H}_e^+ := eH_{\mathfrak{R}}$ and $\mathcal{H}_e^- := ie^\perp H_{\mathfrak{R}}$ (Azizov and Iokhvidov, 1989).

Let $p = p(x, v) \in P$. Put $e = F_x$ and $J_1 = p - p^\perp$ $(=2p - I)$. According to the theory developed in Azizov and Iokhvidov (1989), the restriction of J_1 to $\mathcal{H}_e$ (with $[\cdot, \cdot]$) is the canonical symmetry with respect to the canonical decomposition $\mathcal{H}_e = p\mathcal{H}_e\,[\dot{+}]\,p^\perp\mathcal{H}_e$.

Let $\mathcal{M}$ be a real W^*-algebra of J-real operators. Let us denote by $\mathcal{N}_e$ $[=\mathcal{N}_e(\mathcal{M})]$ the set $\{B \in (\mathcal{M} + i\mathcal{M})\colon B\mathcal{H}_e \subseteq \mathcal{H}_e\}$. Obviously $\mathcal{N}_e$ is a real closed in the strong operator topology *-algebra. In addition, $B \in \mathcal{N}_e$ implies $B^0 \in \mathcal{N}_e$. $\mathcal{N}_e$ is a $W^*\bar{J}$-algebra in the real indefinite metric space $\mathcal{H}_e$. Put $P_e := P \cap \mathcal{N}_e$. Clearly, P_e is a quantum sublogic of P. The logic P_e is called a *hyperbolic* logic.

Let us denote by P_e^+ (P_e^-) the set of all $p \in P_e$ for which the subspace $p\mathcal{H}_e$ is *positive* (i.e., $\forall z \in p\mathcal{H}_e$, $p \neq 0$, $[z, z] > 0$), respectively, *negative* (i.e., $\forall z \in p\mathcal{H}_e$, $z \neq 0$, $[z, z] < 0$). We will denote by $P_e^\pm$ the set P_e^+ or P_e^-. Note that $p \in P_e^+ \Leftrightarrow \bar{J}p \geq 0$ on $\mathcal{H}_e$ and $p \in P_e^- \Leftrightarrow \bar{J}p \leq 0$. For instance, $p(x, v) \in P_{F_e}^+$ and $p(x, v) \in P_{F_x^\perp}^-$. Let $\mathcal{N} = B(H)$. Then the projection $p_y = (\cdot, Jy)y \in P_e^+[(\cdot, Jy)y \in P_e^-] \Leftrightarrow y_{\mathfrak{R}} \in eH_{\mathfrak{R}}$ and $y_{\mathfrak{I}} \in e^\perp H_{\mathfrak{R}} \Leftrightarrow y_{\mathfrak{R}} \in$

$e^{\perp}H_{\Re}$ and $y_{\Im} \in eH_{\mathbb{R}}$). Every projection (Azizov and Iokhvidov, 1989) $p \in P_e$ is representable (not unique!) in the form $p = p_+ + p_-$, where $p_{\pm} \in p_e^{\pm}$.

Some Propositions on Projections in an Indefinite Metric Space

Let $\mathcal{H}$ be a Hilbert space (real or complex) with respect to the Hilbert product $(\cdot, \cdot)$. Let Q^+ $(0 < Q^+ < I)$ be an orthogonal projection on $\mathcal{H}$ and $\mathcal{T} := 2Q^+ - I$, $Q^- := I - Q^+$. Fix the product $[x, y] := (\mathcal{T}x, y)$. $\forall x, y \in \mathcal{H}$. $\mathcal{H}$ is an indefinite metric space with the inidefinite metric $[\cdot, \cdot]$ and with the canonical symmetry $\mathcal{T}$. A W^*-algebra **A** (probably real) acting in $\mathcal{H}$ is called a *$W^*\mathcal{T}$-algebra* if $\mathcal{T} \in$ **A**. A $W^*\mathcal{T}$-algebra **A** is said to be a W^*K-algebra if projections Q^+ and Q^- are infinite with respect to **A**. Let $\mathcal{P}$ $(\mathcal{P}^+, \mathcal{P}^-)$ denote the set of all $\mathcal{T}$-self-adjoint (positive, negative) projections from **A**.

Let Q_1^+ be a maximal positive projection (Azizov and Iokhvidov, 1989) and $\mathcal{T}_1 := 2Q_1^+ - I$. Put $(x, y)_1 := [\mathcal{T}_1 x, y]$ $\forall x, y \in \mathcal{H}$. By the definition, $\mathcal{T}\mathcal{T}_1$ (≥ 0) is an invertible operator. Hence there exist $\alpha, \beta \in \mathbf{R}$ such that $\alpha l \leq \mathcal{T}\mathcal{T}_1 \leq \beta I$. This means that

$$\alpha(x, x) \leq (\mathcal{T}\mathcal{T}_1 x, x) = (x, x)_1 \leq \beta(x, x), \qquad \forall x \in \mathcal{H} \tag{2}$$

Also we have

$$\begin{aligned} |(x, x) - (x, x)_1| &= |((I - \mathcal{T}\mathcal{T}_1)x, x)| \leq \|I - \mathcal{T}\mathcal{T}_1\| \, \|x\|^2 \\ &= \|\mathcal{T} - \mathcal{T}_1\| \, \|x\|^2 = 2\|Q^+ - Q_1^+\| \, \|x\|^2, \qquad \forall x \in \mathcal{H} \end{aligned} \tag{3}$$

Let $p \in \mathcal{P}^+$, $x = Q^+pQ^+$, and $Q^-pQ^+ = w|Q^-pQ^+|$ be the polar decomposition for Q^-pQ^+. The formula

$$p = x + w(x^2 - x)^{1/2} - (x^2 - x)^{1/2} w^* - w(x - F_x)w^*$$

is an indefinite analogy for (1) with a similar proof.

Assume that $p \in \mathcal{P}^+$ is a simple projection in $\mathcal{H}$ with product $(\cdot, \cdot)_1$, i.e., $p = te + (t^2 - t)^{1/2}(w - w^*) - (t - 1)ww^*$; here $t > 1$, $e \leq Q_1^+$, and w is a partial isometry with the initial projection e and final one F_w, $F_w \leq I - Q_1^+$ in $\mathcal{H}$ with $(\cdot, \cdot)_1$. It is clear that $pe - te = (t^2 - t)^{1/2} w$, $ep - te = -(t^2 - t)^{1/2}w^*$. Hence we may identify the minimal real ^{0}-algebra $\mathcal{M}(e, p)$ generated by e and p with $M_2(R)$ (= the algebra of two-by-two real matrices). The algebra $\mathcal{M}(e, p)$ is called a *$W^0\mathcal{T}$-factor of type I_2*.

Let $e, f \in$ **A** be orthogonal projections. We write $e \preceq f$ if there exists a partial isometry $w \in$ **A** with the initial projection e and the final one $F_w \leq f$. We denote by e_p the orthogonal projection onto $\overline{Q^+pH}$, $p \in \mathcal{P}$.

The following result will be needed in Section 4.

Lemma 7. Let $p_n \in \mathcal{P}^+$ and $e_{p_n} \preceq Q^- \wedge (F_{p_n} \vee e_{p_n})^{\perp}$. Then there exists a simple projection $g_n \in \mathcal{P}^+$ such that:

1. $e_{p_n} = e_{g_n}$, $\|e_{g_n} - g_n\| \leq \|e_{p_n} - p_n\|$.
2. Let $Q_1^+ \in \mathcal{P}$ be a maximal positive projection such that $p_n \leq Q_1^+$, and let $\mathcal{T}_1 := 2Q_1^+ - I$. In $\mathcal{H}$ with the Hilbert product $(\cdot, \cdot)_1 := (\mathcal{T}_1 \cdot, \cdot)$ the projection g_n is simple and $Q_1^+ g_n \mathcal{H} = p_n \mathcal{H}$, $\|g_n - p_n\|_1 \leq \|e_{p_n} - p_n\|$.

Proof. We need the index n in p_n, g_n only in the proof of Lemma 12. Hence we do not used the index n in the proof below.

Let $p = (:= p_n) = p(x, v)$ and $\alpha := \frac{1}{2}(\|p\| + 1)$ $(= \|x\|)$. It is clear that $e_p < x \leq \alpha e_p$. One can suppose that $Q^+\mathcal{H} \cap p\mathcal{H} = 0$. Put

$$y_0 := (\alpha - 1)^{-1} (x - e_p) \{\alpha^{1/2} I + [\alpha e_p - (x - e_p)]^{1/2}\}^{-2}$$

Thus $0 < y_0 \leq e_p$. By the assumption, there exists a partial isometry $w \in \mathbf{A}$ with the initial projection vv^* and the final one $F_w \leq Q^- \wedge (F_p \vee e_p)^\perp$. Let

$$z := vy_0^{1/2}v^* + w(F_v - vy_0^{1/2}v^*)^{1/2} \; [= vy_0^{1/2}v^* + wv(e_p - y_0)^{1/2}v^*]$$

It can be easily shown that z is a partial isometry with the initial projection $vv^* = F_v$. By the construction, $g := p(\alpha e_p, zv)$ is a simple projection, $e_g = e_p$ and $\|e_g - g\| = \|e_p - p\|$. The operator $y_0^{1/2}$ is a solution of the equation

$$\alpha(x - e_p)^{1/2} = 2[\alpha(\alpha - 1)]^{1/2} y^{1/2} - (\alpha - 1)(x - e_p)^{1/2} y$$

Making use of this, we can verify that

$$pgp = p(x, v)p(\alpha e_p, zv)p(x, v) = \alpha p(x, v) \tag{4}$$

By (2), the new Hilbert product $(x, y)_1 := [\mathcal{T}_1 x, y]$ is equivalent to $(\cdot, \cdot)$ in $\mathcal{H}$. By (4), $p(\alpha e_p, zv)$ is simple in $\mathcal{H}$ with $(\cdot, \cdot)_1$. By the construction, $\|p - g\|_1 = \|e_p - p\|$. The lemma is proved.

4. MEASURES ON THE LOGIC *P*

Let $(p_i)_{i \in I} \subset P$ be a set of mutually orthogonal projections. Assume that for every subset $X \subseteq I$ there exists $q = \sum_{i \in X} p_i$ (the sum being understood in the strong sense). Then a representation $p = \sum_{i \in I} p_i$ is said to be a *decomposition* of p. Since $p^* = JpJ$, $\forall_p \in P$, we conclude that $p^* = \sum_{i \in I} p_i^*$ is a decomposition of p^* if $p = \sum_{i \in I} p_i$ is a decomposition of p.

A mapping μ: $P \to \mathbf{R}$ is said to be a *measure* (=*quantum measure*) if $\mu(p) = \sum \mu(p_i)$ for any decomposition $p = \sum p_i$. Note that if μ is a measure, then μ^*: $P \to \mathbf{R}$, where $\mu^*(p) := \mu(p^*)$, $\forall_p \in P$, is a measure also.

Let $|||\mu|||_\beta(e) := \sup\{|\mu(g)|: g \in P_\beta, g \leq e\}$, $\beta \geq 1$, $e \in \Pi$, and let $\alpha_\phi^\beta(e) := \sup\{|\mu(p)|(\phi(S_p))^{-1}: p \in P_\beta, p \leq e\}$, $\beta \geq 1$, $e \in \Pi$, where ϕ is a faithful normal semifinite weight on $\mathcal{N}^+$.

A measure μ is said to be *bounded* if $\sup\{\|p\|^{-1}|\mu(p)|: p \in P, p \neq 0\} < +\infty$; ϕ-bounded if $\alpha_\phi^\beta(I) < +\infty$, $\forall\beta \geq 1$; *finite* if $|||\mu|||_\beta(I) < +\infty$, $\forall\beta \geq 1$; *Hermitian* if $\mu(p) = \mu(p^*)$, $\forall p \in P$; *skew Hermitian* if $\mu(p) = -\mu(p^*)$, $\forall p \in P$; *regular* if there exists an operator A such that $\mu(p) = \Re\mathrm{tr}(Ap)$, $\forall p \in P$. Note that a measure μ is the sum $\mu = 1/2(\mu + \mu^*) + 1/2(\mu - \mu^*)$ of Hermitian and skew Hermitian measures.

A trivial computation on two-dimensional matrices shows that the following lemma is true.

Lemma 8. Let $\mu_n(\cdot) = \mathrm{tr}(A_n\cdot)$ be a family of measures in an indefinite metric space K, $\dim K = 2$, with a canonical symmetry $\overline{\mathcal{T}}$. Assume $\alpha_c := \sup\{|\mu_n(p)|: n \in \mathbf{N}, \|p\| \leq c\} < +\infty$. Then for any $\epsilon > 0$ there exists $\delta \in (0, 1)$ such that $p_f \in P^\pm$, $\|p_f\| < 1 + \delta$ implies $\sup_n\{|\mu_n(Q^\pm) - \mu_n(p_f)|\} < \epsilon$. Here $\mathbf{Q}^\pm = (1/2)(I \pm \overline{\mathcal{T}})$.

Lemma 9. Let $\mathcal{N}$ be a semifinite von Neumann algebra and let τ be a faithful normal semifinite trace on $\mathcal{N}^+$. If ν is a τ-bounded measure, then ν is a finite measure:

Proof. First we consider the restriction of ν on P_1 $(= \Pi)$. If $\mathcal{N}$ is a finite von Neumann algebra, then it is clear that $\alpha_\tau^1(I) < +\infty$.

Let $\mathcal{N}$ be a properly infinite von Neumann algebra. Dorofeev (1992) proved that any measure $\mu: \Pi' \to R$ on the set of all orthogonal projections Π' from a von Neumann algebra containing no finite central summands of type I is bounded, i.e., $\sup\{|\mu(p)|: p \in \Pi'\} < +\infty$. It is easily to show by analogy that the measure ν on the set of all (orthogonal) projections from JW-algebra $\mathcal{M}$ containing no finite central summand of type I is bounded also.

Let now $\beta > 1$. It is clear that there exists $e \in \Pi$ such that $\tau(e) < +\infty$ and $|||\nu|||_\beta(e^\perp) < +\infty$. Let $p = p(x, v) \in P_\beta$. Without loss of generality we can assume that p is a properly skew projection.

Let $f := F_p \wedge e^\perp$. By Remark 4(*i*), there exists $g \in P_\beta$ such that $F_g = f$ (and hence $g \leq p$). By the construction of projections from P, $g \leq e^\perp$. By the choice of f, we have $\tau(p - q) = \tau(F_p - f) \leq \tau(e)$. Thus

$$|\nu(p)| \leq |\nu(g)| + |\nu(p - g)| \leq |||\nu|||_\beta(e^\perp) + 2\alpha^{2\beta}\tau(e)$$

The proof is complete.

Lemma 10. Let $\mathcal{N}$ be continuous, countably decomposable von Neumann algebra and let $\nu: P \to \mathbf{R}$ be a measure. For any $\beta \geq 1$ there exists a sequence $\{e_n\} \subset \Pi$, $e_n \to I$ in the strong operator topology such that $|||\nu|||_\beta(e_n) < +\infty$, $\forall n$.

Proof. Let ϕ be a faithful normal state on $\mathcal{N}$. We can assume that $|||\nu|||_\beta(I) = \infty$. Then there exists $p = p(x, v) \in P_\beta$ such that $|\nu(p)| \geq 2^n$. Let $\{p_i\}_1^m$, where $m = 2^n$, be a family from Lemma 5. By the construction, $\phi(S_{p_i}) \leq$

2^{-n}, $\forall i$. It is clear that there exists p_i (without loss of generality we can assume $p_i = p_n$) such that $|\nu(p_n)| \geq 1$.

Let $|||\nu|||_\beta(I - S_{p_n}) = \infty$. By analogy, there exists $p_{n+1} \in P_\beta$, $p_{n+1} \leq I - S_{p_n}$ such that $|\nu(p_{n+1})| \geq 1$ and $\phi(S_{p_{n+1}}) \leq 2^{-(n+1)}$.

We shall continue this process. Let us suppose for the moment that there exists a countable family $\{p_i\}_n^\infty$. Then by the construction, there exists $\Sigma_n^\infty\, p_i \in P_\beta$ and at the same time $|\nu(p_i)| \geq 1$, $\forall_i$. This is a contradiction with the definition of a measure. Therefore there exists a finite family $\{p_i\}_n^k$. Put $e_n := I - \Sigma_n^k\, S_{p_i}$. Then $\phi(e_n) \geq 1 - \Sigma_n^\infty\, 2^{-i}$ and $|||\nu|||_\beta(e_n) < +\infty$. We obtain a suitable family, which completes the proof of Lemma 10.

We are thus led to the following strengthening of Lemma 10.

Lemma 11. Let $\mathcal{N}$ be a W^*J-algebra of type II. Then there exists a sequence $\{r_n\} \subset \Pi$, $r_n \uparrow I$, such that $\tau(r_n^\perp) \downarrow 0$ and $\alpha_\tau^\beta(r_n) < +\infty$, $|||\nu|||_\beta(r_n) < +\infty$, $\forall\beta$ and $\forall n$.

Proof. There exists a unique self-adjoint J-reality operator $A \in L_1(\mathcal{N}, \tau)$ such that $\nu(e) = \tau(A \circ e)$, $\forall e \in \Pi$. Let $A = \int \lambda df_\lambda$ be the spectral representation for A and let $f^n := f_n - f_{-n}$. Let $M(t, e)$ ($e \in \Pi$, $t > 0$) denote a maximal set $\{g_i\} \in P_\beta$, $g_i \leq e$, with mutually orthogonal projections $\{S_{g_i}\}$ such that $\nu(g_i) > tr(S_{g_i})$.

1. Suppose for the moment that $\alpha_\tau^\beta(f^n - (\Sigma\, S_{g_i}\colon g_i \in M(t, f^n)) > t + n$. Then $|\nu(p)|/\tau(S_p) > t + n$, for some $p \leq f^n - (\Sigma\, S_{g_i}\colon g_i \in M(t, f^n))$.

(i) If $\nu(p) > 0$, then $p \in M(t, f^n)$. This is a contradiction with the maximality of $M(t, f^n)$.

(ii) Let $\nu(p) < 0$. We have

$$\nu(p) + \nu(S_p - p) = \nu(S_p) \qquad \text{and} \qquad |\nu(S_p)|\tau(S_p)^{-1} \leq n$$

Hence

$$\frac{\nu(S_p - p)}{\tau(S_p)} = \frac{\nu(S_p)}{\tau(S_p)} - \frac{\nu(p)}{\tau(S_p)} > \frac{\nu(S_p)}{\tau(S_p)} + t + n \geq t$$

In this case $S_p - p \in M(t, f^n)$. We have a contradiction with the maximality of $M(t, f^n)$ again.

Thus

$$\alpha_\tau^\beta(f^n - (\sum S_{g_i}\colon g_i \in M(t, f^n)) \leq t + n \tag{5}$$

2. Fix $\epsilon > 0$. Let us demonstrate that there exists t such that $\tau(\Sigma\, S_{g_i}\colon g_i \in M(t, f^n)) < \epsilon$. Let $m_1 \in \mathbf{N}$ be such that $m_1^{-2} < \epsilon$. Then

$$(1_1) \qquad \tau(\sum S_{g_i}\colon g_i \in M(m_1^3, f^n)) < \epsilon$$

or

$$(2_1) \qquad \tau(\sum S_{g_i}: g_i \in M(m_1^3, f^n)) \geq \epsilon$$

If (1_1), then (5), where $t = m_1^3$.

If (2_1), then there exists $p_1 \in P_\beta$ such that $p_1 \leq f^n$, $\nu(p_1) \geq m_1$, and $\tau(S_{p_1}) \leq m_1^{-2} < \epsilon$. Let $e_1 := S_{p_1}$ and let $m_2 \in \mathbf{N}$ be such that $m_2^{-2} < \epsilon_1 := \epsilon - m_1^{-2}$. Then

$$(1_2) \qquad \tau(\sum S_{g_i}: g_i \in M(m_2^3, f^n - e_1)) < \epsilon_1$$

or

$$(2_2) \qquad \tau(\sum S_{g_i}: g_i \in M(m_2^3, f^n - e_1)) \geq \epsilon_1$$

again.

If (1_2), then

$$\alpha_\tau^\beta\ (f_n - e_1 - (\sum S_{g_i}: g_i \in M(m_2^3, f^n - e_1))) \leq m_2^3 + n$$

and

$$\tau((\sum S_{g_i}: g_i \in M(m_2^3, f^n - e_1)) + e_1) < \epsilon_1 + m_1^{-2} < \epsilon$$

If (2_2), then there exists $p_2 \in P_\beta$ such that $p_2 \leq f^n - e_1$, $\nu(p_2) \geq m_2$, and $\tau(S_{p_2}) \leq m_2^{-2}$.

If we continue this process, then the process (2_n) stops at some step k. Otherwise we have the sequence $\{p_n\}_1^\infty \in P_\beta$ of mutually orthogonal projections with the property $\nu(p_n) > m_n$ and $p := \Sigma\, p_n \in P_\beta$, contradicting the definition of the measure.

Thus the inequality

$$(1_k) \qquad \tau((\sum S_{g_i}: g_i \in M(m_k^3, f^n - e_1 - e_2 - \cdots - e_{k-1})) < \epsilon_{k-1}$$

is true. By the construction,

$$\tau((\sum S_{g_i}: g_i \in M(m_k^3, f^n - e_1 - \cdots - e_{k-1})) + e_1 + \cdots + e_{k-1}) < \epsilon$$

Hence for a given $\epsilon = 2^{-n}$ there exists $e_n(\beta) \in \Pi$ with the properties $e_n(\beta) \leq f^n$, $\tau(e_n(\beta)) < 2^{-n}$, and $\alpha_\tau^\beta(f^n - e_n(\beta)) < +\infty$. Let n_k be such that $\tau(I - f^{n_k}) < 2^{-k}$. Let $\beta = m$. By Lemma 9, the sequence $r_n := \wedge_{m \geq n}(f^{n_m} - e_{n_m}(m))$ is suitable.

Let $p(x, v) \in P_\beta$. Put $P_{x,v}^\beta = \{p(x_o, v_o) \in P_\beta: F_{x_o} \leq F_x\}$.

Lemma 12. Let $\mathcal{N}$ be a W^*J-algebra containing no central summand of type I_2. Let the projection $p(x, v)$ and a measure ν be such that:

(i) $\sup\{|\nu(p(x_o, v_o))|: p(x_o, v_o) \in P_{x,v}^\beta\} < +\infty$, $\forall \beta \geq 1$.

(ii) The restriction of ν on any $W^0\mathcal{T}$-subfactor of type I_2 is a regular measure.

Then $\nu(p(x, v)) = \lim \nu(p(x_n, v))$, $\forall p(x, v) \in P$. Here $p(x_n, v)$ is the sequence from Corollary 2.

Proof. Let $p := p(x, v) \in P$. Put $e := F_x$. Let $\mathcal{H}_e$ be the Hilbert space, $\mathcal{N}_e$ be the algebra, $[\cdot, \cdot]$ be the indefinite metric, and the logic P_e be as in Section 3. By the construction, $p(x, v)$ and $p(x_n, v)$ ($\in P_e$) are maximal positive projections. Let $(x, y)_0 := [(2p(x, v) - I)x, y]$ be a new Hilbert product in $\mathcal{H}_e$. With respect to the $(\cdot, \cdot)_0$ the operator $p(x, v)$ is an orthogonal projection. Let $Q^+ := p(x, v)$, and $Q^- := I - Q^+$, and $p_n := p(x_n, v)$.

1. Let first the pair (p, p_n) $\forall n$ be such that $e_{p_n} \preceq Q^- \wedge (F_{p_n} \vee e_{p_n})^\perp$ in $\mathcal{H}_e$ with $(\cdot, \cdot)_0$. [By the definition of Q^+ and $p(x_n, v)$, we have $e_{p_n} = Q^+$ in $\mathcal{H}_e$ with $(\cdot, \cdot)_0$.] Let g_n be the projection from Lemma 7. By the construction of $\mathcal{H}_e$, we have $p_n, g_n \in P^\beta_{x,v}$ $\forall_n \in \mathbf{N}$, and some $\beta > 1$. By the construction again, the minimal $-^0$ algebras $\mathcal{N}(p, g_n)$ and $\mathcal{N}(p_n, g_n)$ generated by p, g_n and p_n, g_n, are $W^0\mathcal{T}$-factors of type I_2.

By the assumption, the restriction of ν on $\mathcal{N}(p, g_n)$ and on $\mathcal{N}(p_n, g_n)$ is a regular measure. Let us identify $\mathcal{N}(p, g_n)$ and $\mathcal{N}(p_n, g_n)$ with the algebra $M_2(R)$ of all two-by-two real matrices so that g_n corresponds to $\mathfrak{Q}^+ = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$. This generates a family μ_n of regular measures on $M_2(R)$. By (i), the sequence μ_n satisfies the assumption of Lemma 8. By the construction, $\|p - g_n\| \to 0$ and $\|g_n - p_n\| \to 0$ $(n \to \infty)$. By Lemma 8, for a given $\epsilon > 0$ there exists $\mathbf{N}(\epsilon)$ such that $n > \mathbf{N}(\epsilon)$ implies

$$|\nu(p) - \nu(p_n)| \leq |\nu(p) - \nu(g_n)| + |\nu(g_n) - \nu(p_n)| < \epsilon$$

2. In the general case there exist decompositions $p = p^1 + p^2 + p^3$ and $p_n = p_n^1 + p_n^2 + p_n^3$ such that $\|p^i - p_n^i\|_{n\to\infty} \to 0$ and the pair (p^i, p_n^i) $(i = 1, 2, 3)$ satisfies step 1. Thus

$$\nu(p) = \sum_1^3 \nu(p^i) = \sum_1^3 \lim \nu(p_n^i) = \lim \nu(p_n)$$

We can now prove our main result.

Theorem 13. Let $\mathcal{M}$ be a real W^*-algebra of J-real bounded operators containing no finite central summand in a complex Hilbert space H with conjugation J and let P be the quantum logic of all J-orthogonal projections in the von Heumann algebra $\mathcal{N} = \mathcal{M} + i\mathcal{M}$. Let $\mu: P \to R$ be a Hermitian measure. Then

$$\mu(p) = \Re\psi(p), \qquad \forall p \in P \tag{6}$$

where ψ is a J-self-adjoint ultraweakly continuous linear functional on $\mathcal{N}$ such that $\mu(p) = \psi(p)$, $\forall p \in \Pi$.

Proof. The proof will consist of several steps.

1. First suppose that $\mathcal{M}$ is the set $B_{\mathcal{J}\mathfrak{R}}(H)$ of all J-real bounded operator in H, dim $H = +\infty$. By Dorofeev and Sherstnev (1990), any measure on the set of all orthogonal projections in an infinite-dimensional Hilbert space is bounded. It is easy to see that any bounded measure is regular. The set II is isomorphic to the set of all orthogonal projections on $H_{\mathfrak{R}}$. Hence for the restriction of μ to Π, there exists a unique J-real self-adjoint trace class operator A such that $\mu(p) = \text{tr}(p)$, $\forall p \in \Pi$.

By Corollary 3 again, it suffices to prove the equality $\mu(p_f) = \mathfrak{R}\text{tr}(Ap_f)$ for any one-dimensional projection $p_f \in P$, where $f = c_R\varphi + ic_I\varphi^\perp$, c_R, $c_I \in R$, $c_R^2 - c_I^2 = 1$, φ, $\varphi^\perp \in S_{\mathfrak{R}}$, and $(\varphi, \varphi^\perp) = 0$. There exists $e \in \Pi$ such that $p_\varphi \leq e$, dim $eH = +\infty$, $p_{\varphi^\perp} \leq e^\perp$, and dim $e^\perp H = +\infty$. Let $\mu_e(p) := \mu(p_H)$, $\forall p \in P_e$. It is clear that μ_e is a measure on P_e. By Theorem 2.1 (Matvejchuk, 1997), there exists a unique trace class operator $B \in B(\mathcal{H}_e)$, $\overline{J}$-self-adjoint in $\mathcal{H}_e$, such that $\mu_e(p) = \text{tr}_{\mathcal{H}_e}(Bp)$, $\forall p \in P_e$. Since $\text{tr}_{\mathcal{H}_e}(Bp) = \mu(p) \in R$, it follows that

$$\text{tr}_{\mathcal{H}_e}(Bp) = \text{tr}_{\mathcal{H}_e}(Bp^*) = \text{tr}_{\mathcal{H}_e}((Bp^*)^*) = \text{tr}_{\mathcal{H}_e}(B^*p)$$

Hence $\mu_e(p) = \text{tr}_{\mathcal{H}_e}(\frac{1}{2}(B + B^*)p)$, $\forall p \in P_e$. Thus we can assume that B is self-adjoint in $\mathcal{H}_e$ and J-self-adjoint, i.e., $B = \overline{J}B\overline{J}$. This means that $B = eBe + e^\perp Be^\perp$. We have

$$\mu(p_f) = \mu_e(p_f) = \text{tr}(Bp_j) = (Bf, Jf)$$

Put $B_\varphi := (p_\varphi + p_{\varphi^\perp})B(p_\varphi + p_{\varphi^\perp})$. Then $B_\varphi = \mu(p_\varphi)p_\varphi + \mu(p_{\varphi^\perp})p_{\varphi^\perp}$ is the spectral decomposition for B_φ. It is obvious that

$$\mu(p_\varphi) = \text{tr}(Ap_\varphi) = a|(\varphi, \kappa)|^2 + b|(\varphi, \kappa^\perp)|^2$$

and

$$\mu(p_{\varphi^\perp}) = a|(\varphi^\perp, \kappa)|^2 + b|(\varphi^\perp, \kappa^\perp)|^2$$

for some κ, $\kappa^\perp \in H_{\mathfrak{R}}$, $(\kappa, \kappa^\perp) = 0$, $a, b \geq 0$.

Then

$$\begin{aligned}\mu(p_f) &= \text{tr}(Bp_f) = (Bf, Jf) = \mu(p_\varphi)(p_\varphi f, Jf) + \mu(p_{\varphi^\perp})(p_{\varphi^\perp} f, Jf)\\ &= \mu(p_\varphi)(c_R\varphi, Jf) + \mu(p_{\varphi^\perp})(ic_I\varphi^\perp, Jf) = \mu(p_\varphi)c_R^2 - \mu(p_{\varphi^\perp})c_I^2\\ &= [a|(\varphi, \kappa)|^2 + b|(\varphi, \kappa^\perp)|^2]c_R^2 - [a|(\varphi^\perp, \kappa)|^2 + b|(\varphi^\perp, \kappa^\perp)|^2]c_I^2\\ &= a(|(\varphi, \kappa)|^2c_R^2 - |(\varphi^\perp, \kappa)|^2c_I^2) + b(|(\varphi, \kappa^\perp)|^2c_R^2 - |(\varphi^\perp, \varphi^\perp)|^2c_I^2)\\ &= \mathfrak{R}(a(f, \kappa)(\kappa, Jf) + b(f, \kappa^\perp)(\kappa^\perp, Jf)) = \mathfrak{R}\text{tr}(Ap_f)\end{aligned}$$

2. Now without loss of generality we can assume that $\mathcal{M}$ contains no central summands of factor type I_∞.

In the proof of Lemma 9 we remarked that $\sup\{|\mu(p)|: p \in \Pi\} < +\infty$. In Matvejchuk (1995) it was proved that a bounded measure on the set of all projections from a *JW*-algebra containing no central summand of type I_2 continued to a linear functional on the *JW*-algebra. This means that there exists a *J*-self-adjoint ultraweakly continuous linear functional ψ such that $\mu(p) = \psi(p)$, $\forall p \in \Pi$. By step 1, we have $\mu(p(x, v)) = \Re\psi(p(x, v))$ if $p(x, v) = \Sigma\, p(t_n e_n, v)$ is a sum of orthogonal family of simple projections from P. Hence we can prove (6) for $p(x, v) \in P$ where x has a continuous spectrum on (c, ∞), where $c = \inf\{(x\kappa, \kappa), \kappa \in S \cap e_+ H\}$. In addition, if $\mathcal{N}$ is a von Neumann algebra of type I_∞, we can assume that $p(x, v)$ is an Abelian projection (i.e., F_p is an Abelian projection). Let $p(x_n, v)$ be the sequence from Corollary 2.

(i) Let $\mathcal{N}$ be a W^*J-algebra of type Π_∞ and let $\{r_n\}$ be the sequence from Lemma 11. By Lemma 11, the restriction of μ onto $\{p \in P: p \leq r_n\}$ is the finite measure. The W^*J-algebra $r_n \mathcal{N} r_n$ acting in $r_n H$ has the infinite type. Using this and Theorem 2.1 (Matvejchuk, 1997), we conclude that the restriction of μ on any W^0J-subfactor $\mathcal{L} \subset r_n \mathcal{N} r_n$ of type I_2 is a regular measure. Let $p(x, v) \in P$, $p(x, v) \leq r_n$. By step 1, $\mu(p(x_n, v)) = \Re\psi(p(x_n, v))$. By Lemma 12,

$$\mu(p(x, v)) = \lim \mu(p(x_n, v)) = \lim \Re\psi(p(x_n, v)) = \Re\psi(p(x, v))$$

It is clear that in the general case $p \in P$ there exists a sequence $\{p_n\} \subset P$ such that $p_n \leq r_n$ and $p_n \uparrow p$. This means that (6) holds.

(ii) Now let $\mathcal{N}$ be a W^*J-algebra of type I_∞. Let $p(x, v) \in P$ be an Abelian projection. For an Abelian $f \in \Pi$, $\tau(f) < +\infty$ there exists a sequence $\{f_m\} \subset \Pi$ such that $f_m | f$ and $\sup\{|\mu(p(x, v))|: p(x, v) \in P_\alpha, F_x \leq f_m\} < \infty$, $\forall m \in \mathbf{N}$.

Let $f = F_x$. Fix $m \in \mathbf{N}$. By step 1 and by Lemma 12,

$$\begin{aligned} \mu(p(xf_m, vf_m)) &= \lim_{n\to\infty} \mu(p(x_n f_m, vf_m)) \\ &= \lim_{n\to\infty} \Re\psi(p(x_n f_m, vf_m)) \\ &= \Re\psi(p(xf_m, vf_m)) \end{aligned}$$

Finally,

$$\begin{aligned} \mu(p(x, v)) &= \lim_{m\to\infty} \mu(p(xf_m, vf_m)) \\ &= \lim_{m\to\infty} \Re\psi(p(xf_m, vf_m)) = \Re\psi(p(x, v)) \end{aligned}$$

It is clear that (6) in the general case of F_x is true, also QED

5. SOME PROPOSITIONS ON BOUNDEDNESS OF MEASURES

We prove that the boundedness of a measure essentially depends on the dimension of H. Let $\dim H < +\infty$. Consider a measure $\mu(p) = \Re\operatorname{tr}(Ap)$, $\forall p \in P$, where $A \neq \lambda I$, and let m be a discontinuous endomorphism of the additive group of the real numbers. Then $m \circ \mu$ is an unbounded (and hence nonregular) measure on P.

Proposition 14. Let $\nu: P \to R$ be a bounded measure on $B(H)$, $\dim H \geq 3$. Then for any $x \in S_{\Re}$ the restriction of ν onto $P_{(\cdot,x)x}$ is a regular measure.

Proof. Fix $x, y, z \in S_{\Re}$, $(x, y) = (y, z) = (z, x) = 0$. By Theorem 2.3 (Matvejchuk, 1997), there exist unique $\bar{J}$-self-adjoint in $\mathcal{H}_{(\cdot,\, x)x}$ trace class operator A'_x and unique number c_x such that

$$\nu(p) = \operatorname{tr}(A'_x p) + c_x \dim p_+ H, \quad \forall p \in P_{(\cdot,x)x}$$

(Note that

$$\nu(p) = \operatorname{tr}(A_x p) - c_x \dim p_- H, \quad \forall p \in P_{(\cdot,x)x} \quad \dim pH < +\infty$$

Here $A_x = A'_x + c_x I$. Note also that $0 \leq \dim p_+ H \leq 1$, $\forall p \in P_{(\cdot,x)x}$).

Thus if we prove that $c_x = 0$, the assertion follows. By analogy, there exist unique $\bar{J}$-self-adjoint operators A_y and A_z in $\mathcal{H}_{(\cdot y)y}$ and $\mathcal{H}_{(\cdot z)z}$, respectively, and unique numbers c_y, c_z such that

$$\nu(p) = \operatorname{tr}(A_y p) - c_y \dim p_- H, \quad \forall p \in P_{(\cdot,y)y}, \quad \dim pH < +\infty$$

$$\nu(p) = \operatorname{tr}(A_z p) - c_z \dim p_- H, \quad \forall p \in P_{(\cdot,z)z}, \quad \dim pH < +\infty$$

It is clear that

$$P_{x,y} := \{p_f \in P: f_{\Re}, f_{\mathfrak{I}} \in \{\lambda x\} \cup \{\beta y\}_{\lambda,\beta \in R}\} \subset P_{(\cdot,x)x} \cap P_{(\cdot,y)y}$$

Let $p_f \in P_{x,y}$. Then

$$\operatorname{tr}(A'_x p_f) + c_x \dim(p_f)_+ H = \nu(p_f) = \operatorname{tr}(A_y p_f) - c_y \dim(p_f)_- H$$

Here $(p_f)_+$ is the positive part of p_f in $P_{(\cdot x)x}$ and $(p_f)_-$ is the negative part of p_f in $P_{(\cdot y)y}$. But $p_f \in P^-_{(\cdot,y)y} \Leftrightarrow p_f \in P^+_{(\cdot,x)x}$. Thus

$$\operatorname{tr}((A_y - A'_x)p_f) = (c_x + c_y) \dim(p_f)_+ H \tag{7}$$

1. The right of (7) is a discontinuous bounded function on $P_{x,y}$ if $c_x + c_y \neq 0$.

Denote by e the projection $(\cdot, x)x + (\cdot, y)y$.

2(i). If $e(A_y - A'_x)e = \lambda e$, then $\operatorname{tr}((A_y - A'_x)p_f) = \lambda$, $\forall p_f \in P_{x,y}$.

2(ii). If $e(A_y - A'_x)e \neq \lambda e$, $\forall \lambda \in \mathbf{R}$, then $p_f \to \operatorname{tr}((A_y - A'_x)p_f)$ is an unbounded function on $P_{x,y}$.

By 1, 2(i) and 2(ii), we have $c_x + c_y = 0$ and $e(A_y - A'_x)e = 0$. This means that $c_y = -c_x$. By analogy, $c_z = -c_x$.

By definition, $P_{y,z} \subset P_{(\cdot,y)y} \cap P_{(\cdot,z)z}$. Let $p_f \in P_{y,z}$. Then $p_f \in P^+_{(\cdot,y)y} \Leftrightarrow p_f \in P^-_{(\cdot,z)z}$. Hence

$$\begin{aligned} &\mathrm{tr}(A_y p_f) + c_x \dim(p_f)_- H = \mathrm{tr}(A_y p_f) - c_y \dim(p_f)_- H = \nu(p_f) \\ &\quad = \mathrm{tr}(A_z p_f) - c_z \dim(p_f)_+ H = \mathrm{tr}(A_z p_f) + c_x \dim(p_f)_+ H, \quad \forall p_f \in P_{y,z} \end{aligned}$$

Here $(p_f)_+$ and $(p_f)_-$ are positive and negative parts of p_f in $P_{(\cdot,y)y}$, respectively. Thus

$$\mathrm{tr}((A_y - A_z)p_f) = c_x(\dim(p_f)_+ H - \dim(p_f)_- H), \qquad \forall p_f \in P_{y,z} \tag{8}$$

We have again:

1. The right of (8) is a discontinuous bounded function on $P_{y,z}$ if $c_x \neq 0$.

Let us consider the left of (8). Now denote by e the projection $(\cdot, y)y + (\cdot, z)z$.

2(i). If $e(A_y - A_z)e = \lambda e$ for some $\lambda \in \mathbf{R}$, $\lambda \neq 0$, then $\mathrm{tr}((A_y - A_z)p_f) = \lambda$, $\forall p_f \in P_{x,z}$.

2(ii). If $e(A_y - A_z)e \neq \lambda e$, $\forall \lambda \in \mathbf{R}$, then the function $p_f \to \mathrm{tr}((A_y - A_z)p_f)$ is unbounded on $P_{x,z}$.

By 1, 2(i), and 2(ii), we have $e(A_y - A_z)e = 0$ and $c_x = 0$. The lemma is proved.

Corollary 15. Let ν: $P \to R$ be a bounded Hermitian measure on $B(H)$, $\dim H \geq 3$. Then $\nu(p) = \Re\mathrm{tr}(Ap)$, $\forall p \in P$, where A is a J-real self-adjoint trace class operator such that $\nu(p) = \mathrm{tr}(Ap)$, $\forall p \in \Pi$.

Proof. By Corollary 3, it suffices to prove that $\nu(p_f) = \Re\mathrm{tr}(Ap_f)$, $\forall p_f \in P$. We remarked in the introduction that Π is isomorphic to the lattice of all orthogonal projections on $H_{\Re}$. Thus, by the boundedness of ν on Π, there exists J-reality self-adjoint trace class operator A such that $\nu(p) = \mathrm{tr}(Ap)$, $\forall p \in \Pi$.

Fix $x, y \in S_{\Re}$, $(x, y) = 0$. Let A_x from the proof of Proposition 14 and $f = ax + iby$, $a^2 - b^2 = 1$. Then

$$\mathrm{tr}(A_x p_f) = \nu(p_f) = \nu(p_f^*) = \mathrm{tr}(A_x p_f^*) = \mathrm{tr}(A_x^* p_f)$$

Thus we can assume that $B := (p_x + p_y)A_x(p_x + p_y)$ is self-adjoint and J-self-adjoint. This means that $B = \alpha p_x + \beta p_y$, where $\alpha = \mathrm{tr}(Bp_x) = \nu(p_x) = \mathrm{tr}(Ap_x)$ and $\beta = \nu(p_y)$. Finally,

$$\begin{aligned}\nu(p_f) &= \mathrm{tr}(A_x p_f) = \mathrm{tr}(Bp_f) = (Bf, Jf) \\ &= \Re(Af, Jf) = \Re\mathrm{tr}(Ap_f), \qquad \forall p_f \in P\end{aligned}$$

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